9.3 How well does a quantum channel preserve information?

– for dynamics distance measurements.
$|\psi\rangle \rightarrow \mathcal{E}(|\psi\rangle\langle\psi|)$

This type of scenario occurs often in quantum computation and quantum information.

eg) we can compute the fidelity between the starting state $|\psi\rangle$ and the ending state $\mathcal{E}(|\psi\rangle\langle\psi|)$. For the case of the depolarizing channel(page 378, (8.100)), we obtain

$$\begin{aligned} F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) &= \sqrt{\langle\psi|\left(p\frac{I}{2} + (1-p)|\psi\rangle\langle\psi|\right)|\psi\rangle} && (9.112) \\ &= \sqrt{1-\frac{p}{2}}. && (9.113) \end{aligned}$$

This result agrees well with our intuition - the higher the probability p of depolarizing, the lower the fidelity of the final state with the initial state. ($\rightarrow$ We could equally well have used the trace distance)

In a real quantum memory or quantum communications channel, we don't know in advance what the initial state $|\psi\rangle$ of the system will be. However, we can quantify the worst-case behavior of the system by minimizing over all possible initial states,

$$F_{\min}(\mathcal{E}) \equiv \min_{|\psi\rangle} F(|\psi\rangle, \mathcal{E}(|\psi\rangle, |\psi\rangle\langle\psi|)). \tag{9.114}$$

For the phase damping channel,

$$\mathcal{E}(\rho) = p\rho + (1-p)Z\rho Z, \tag{9.115}$$

the fidelity is given by

$$\begin{aligned} F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) &= \sqrt{\langle\psi|(p|\psi\rangle\langle\psi| + (1-p)Z|\psi\rangle\langle\psi|Z)|\psi\rangle} && (9.116) \\ &= \sqrt{p + (1-p)\langle\psi|Z|\psi\rangle^2}. && (9.117) \end{aligned}$$

The second term is

$$0 \leq \langle\psi|Z|\psi\rangle^2 \leq 1. \;\; (\; |\psi\rangle = |+\rangle \rightarrow \langle Z\rangle = 0 \;)$$

Thus for the phase damping channel the minimum fidelity is

$$F_{\min}(\mathcal{E}) = \sqrt{p}. \tag{9.118}$$

You might wonder why we minimized over *pure states* in the definition of $F_{\min}$. Might not the quantum system of interest start in a mixed state ρ? Fortunately, the joint concavity of the fidelity can be used to show that allowing mixed states can be used to show that allowing mixed states does not change $F_{\min}$.

Suppose that $\rho = \sum_i \lambda_i |i\rangle\langle i|$ is the initial state of the quantum system. Then we have

$$\begin{aligned} F(\rho, \mathcal{E}(\rho)) &= F\left(\sum_i \lambda_i |i\rangle\langle i|, \sum_i \lambda_i \mathcal{E}(|i\rangle\langle i|)\right) && (9.119) \\ &\geq \sum_i \lambda_i F(|i\rangle, \mathcal{E}(|i\rangle\langle i|)). && (9.120) \end{aligned}$$

It follows that

$$F(\rho, \mathcal{E}(\rho)) \geq F(|i\rangle, \mathcal{E}(|i\rangle\langle i|)) \tag{9.121}$$

for at least one of the states $|i\rangle$, and thus $F(\rho, \mathcal{E}(\rho)) \geq F_{\min}$

Suppose that as part of a quantum computation we attempt to implement a quantum gate described by the unitary operator U. (the operation will have inevitably some noise) The correct description of the gate is using a trace-preserving quantum operation $\mathcal{E}$.A natural measure of how successful our gate has been is the *gate fidelity*,

$$F(U,\mathcal{E}) \equiv \min_{|\psi\rangle} F(U|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)). \tag{9.122}$$

eg) A NOT gate on a single qubit, but instead implement the noisy operation

$$\mathcal{E}(\rho) = (1-p)X\rho X + pZ\rho Z$$

. Then the gate fidelity is given by

$$\begin{aligned} F(X,\mathcal{E}) &= min_{|\psi\rangle}\sqrt{\langle\psi|(X(1-p)X|\psi\rangle\langle\psi|X + pZ|\psi\rangle\langle\psi|Z)X)|\psi\rangle} && (9.123)\\ &= min_{|\psi\rangle}\sqrt{(1-p) + p\langle\psi|Y|\psi\rangle^2} && (9.124)\\ &= \sqrt{1-p} && (9.125)\end{aligned}$$

Quantum sources of information and the entanglement fidelity

Quantum sources of information;

- a ensemble : a stream of identical quantum systems being produce by some physical process
- entanglement : motivated by the idea that a channel which preserves information well is a channel that preserves entanglement

Thus those two sources lead to notions of *ensemble average fidelity*, and *entanglement fidelity.*

The ensemble fidelity captures the idea that the source is well-reserved under the action of a noisy channel described by a trace-preserving quantum operation $\mathcal{E}$, namely

$$\bar{F} = \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2, \tag{9.126}$$

where p_j are the respective probabilities for the different possible preparations of the system ρ_j. ($0 \leq \bar{F} \leq 1$) $F(\rho_j, \mathcal{E}(\rho_j))^2 \rightarrow$ it is not entirely clear what the 'correct'definitions for notions such as information preservation are.

The entanglement fidelity is from the discussion of the classical probability of error in Section 9.1. ($\rightarrow$ there is no direct quantum analogue of a probability distribution defined at two times.

A quantum system, Q, is prepared in a state ρ. The state of Q is assumed to be *entangled* in some way with the external world. This entanglement replaces the (quantum) correlation between X and $\tilde{X}$ in the classical model. We represent the entanglement by introduction a fictitious system R, such that the joint state of RQ is a pure state. It turns out that all results that we prove do not depend in any way on how this purification is performed, so we may as well suppose that this is an arbitrary entanglement with the outside world. The system Q is then subjected to a dynamics described by a quantum operation, $\mathcal{E}$.

The entanglement fidelity $F(\rho,\mathcal{E})$ is

$$\begin{aligned} F(\rho,\mathcal{E}) &\equiv F(RQ, R'Q')^2 && (9.127)\\ &= \langle RQ|[(\mathcal{I}_R\otimes\mathcal{E})(|RQ\rangle\langle RQ|)]|RQ\rangle, && (9.128)\end{aligned}$$

where the use of a prime indicates the state of a system after the quantum operation has been applied, and the absence of a prime indicates the state of a system before the quantum operation has been applied. **The use of the square of the static fidelity is purely a convenience**. The entanglement fidelity depends only upon ρ and $\mathcal{E}$. To see this, use $|R_2Q_2\rangle = U|R_1Q_1\rangle$. Thus

$$F(|R_2Q_2\rangle, \rho^{R_2'Q_2'}) = F(|R_1Q_1\rangle, \rho^{R_1'Q_1'}). \tag{9.129}$$

The entanglement fidelity provides a measure of how well the entanglement between R and Q is preserved by the process $\mathcal{E}$.

Suppose E_i is a set of operation elements for a quantum operation $\mathcal{E}$. Then

$$F(\rho,\mathcal{E}) = \langle RQ|\rho^{R'Q'}|RQ\rangle = \sum_i |\langle RQ|E_i|RQ\rangle|^2. \tag{9.130}$$

Suppose we write $|RQ\rangle = \sum_j \sqrt{p_j}|j\rangle|j\rangle$, where $\rho = \sum_j p_j |j\rangle\langle j|$. Then

$$\begin{aligned}
\langle RQ|E_i|RQ\rangle &= \sum_{jk} \sqrt{p_j p_k}\langle j|k\rangle\langle j|E_i|k\rangle && (9.131)\\
&= \sum_j p_j \langle j|E_i|j\rangle && (9.132)\\
&= \mathrm{tr}(E_i\rho). && (9.133)
\end{aligned}$$

Thus

$$F(\rho,\mathcal{E}) = \sum_i |\mathrm{tr}(\rho E_i)|^2. \tag{9.134}$$

eg) The entanglement fidelity for the phase damping channel (phase flip)

$$\mathcal{E}(\rho) = (1-p)\rho + (1-p)Z\rho Z$$

is given by

$$F(\rho,\mathcal{E}) = (1-p)|\mathrm{tr}(\rho)|^2 + p|\mathrm{tr}(\rho Z)|^2 = (1-p) + p\,\mathrm{tr}(\rho Z)^2, \tag{9.135}$$

so we see that as p increases, the entanglement fidelity decreases.

These two entanglements, that is, ensemble fidelity and entanglement fidelity are closely related. The entanglement fidelity is a lower bound on the square of the static fidelity between the input and output to the process,

$$F(\rho,\mathcal{E}) \le [f(\rho,\mathcal{E}(\rho))]^2. \tag{9.136}$$

Intuitively, this result states that it is harder to preserve a state plus entanglement with the outside world than it is to merely preserve the state alone.

The second property of entanglement fidelity we need to relate it to the ensemble average definition is that it is a *convex* function of ρ. Define $f(x) \equiv F(x\rho_1 + (1-x)\rho_2, \mathcal{E})$, and using (9.135)

$$f''(x) = \sum_i |\mathrm{tr}((\rho_1-\rho_2)E_i)|^2, \tag{9.137}$$

so that $f''(x) \ge 0$.

Combing these two results we see that

$$\begin{aligned}
F\left(\sum_j p_i\rho_j, \mathcal{E}\right) &\le \sum_j p_j F(\rho_j,\mathcal{E}) && (9.138)\\
&\le \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2, && (9.139)
\end{aligned}$$

and thus

$$F\left(\sum_j p_i\rho_j, \mathcal{E}\right) \le \bar{F}. \tag{9.140}$$

Thus, any quantum channel $\mathcal{E}$ which does a good job of preserving the entanglement between a source described by a density operator ρ and a reference system will automatically do a good job of preserving an ensemble source described by probabilities p_j and stats ρ_j such that $\rho = \sum_j p_j \rho_j$. $\rightarrow$ a quantum source based on entanglement fidelity is a more stringent notion than the ensemble definition, and for this reason **the entanglement fidelity based definition in our study of quantum information theory**

Properties of the entanglement fidelity

1. $0 \leq F(\rho, \mathcal{E}) \leq 1$.
2. The entanglement fidelity is linear in the quantum operation input ($F = \langle\psi|\rho|\psi\rangle$)
3. For pure state inputs, the entanglement fidelity is equal to the static fidelity squared between input and output
$$F(|\psi\rangle, \mathcal{E}) = F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|))^2. \tag{9.141}$$
4. $F(\rho, \mathcal{E}) = 1$ iff for all pure states $|\psi\rangle$ in the support of ρ,
$$\mathcal{E}(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|. \tag{9.142}$$
5. Suppose that $\langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle \geq 1 - \eta$ for all $|\psi\rangle$ in the support of ρ, for some η. Then $F(\rho, \mathcal{E}) \geq 1 - (3\eta/2)$.