9.2 How close are two quantum state?

9.2.1 Trace distance

We begin by defining the *trace distance* between quantum states ρ and σ,

$$D(\rho,\sigma) \equiv \frac{1}{2}\mathrm{tr}|\rho-\sigma|, \tag{9.11}$$

where as per usual we define $|A| \equiv \sqrt{A^\dagger A}$.

IF ρ and σ commute, quantum trace distance between ρ and σ is equal to the classical distance between the eigenvalues of ρ and σ.

$$\rho = \sum_i r_i|i\rangle\langle i|; \quad \sigma \sum_i s_i|i\rangle\langle i|. \tag{9.12}$$

Thus

$$\begin{aligned} D(\rho,\sigma) &= \frac{1}{2}\mathrm{tr}\left|\sum_i (r_i-s_i)|i\rangle\langle i|\right| & (9.13)\\ &= D(r_i,s_i) & (9.14)\end{aligned}$$

A good way of getting a feel for the trace distance is to understand it for the special case of a qubit, in the Bloch sphere representation. Suppose ρ and σ have respective Bloch vectors $\vec{r}$ and $\vec{s}$,

$$\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}; \quad \sigma = \frac{I+\vec{s}\cdot\vec{\sigma}}{2}. \tag{9.17}$$

The trace distance between ρ and σ is easily calculated:

$$\begin{aligned} D(\rho,\sigma) &= \frac{1}{2}\mathrm{tr}|\rho-\sigma| & (9.18)\\ &= \frac{1}{4}\mathrm{tr}|(\vec{r}-\vec{s})\cdot\vec{\sigma}|. & (9.19)\end{aligned}$$

$(\vec{r}-\vec{s})\cdot\vec{\sigma}$ has eigenvalues $\pm|\vec{r}-\vec{s}|$, so the trace of $|(\vec{r}-\vec{s})\cdot\vec{\sigma}|$ is $|\vec{r}-\vec{s}|$, and we see that

$$D(\rho,\sigma) = \frac{|\vec{r}-\vec{s}|}{2}. \tag{9.20}$$

That is, the distance between two single qubit states is equal to one half the ordinary Euclidean distance between them on the Bloch sphere.

eg) rotations of the Bloch sphere leave the Euclidean distance invariant

$$D(U\rho U^\dagger, U\sigma U^\dagger) = D(\rho,\sigma). \tag{9.21}$$

The trace distance generalization is

$$D(\rho,\sigma) = \max_P \mathrm{tr}\,[2]\,(P(\rho-\sigma)), \tag{9.22}$$

where the maximization may be taken alternately over all projectors, P, or over all positive operators $P \leq I$.

$$\rho - \sigma = Q - S,$$

where Q and S are positive operators with orthogonal support. It implies that

$$|\rho-\sigma| = Q + S,$$

so

$$D(\rho,\sigma) = \frac{1}{2}(\mathrm{tr}(Q) + \mathrm{tr}(S)).$$

But

$$\mathrm{tr}(Q - S) = \mathrm{tr}(\rho - \sigma) = 0,$$

so

$$\mathrm{tr}(Q) = \mathrm{tr}(S)$$

and therefore

$$D(\rho,\sigma) = \mathrm{tr}(Q).$$

Let P be the projector onto the support of Q. Then

$$\begin{aligned}\mathrm{tr}(P(\rho-\sigma)) &= \mathrm{tr}(P(Q-S)) \\ &= \mathrm{tr}(Q) \\ &= D(\rho,\sigma).\end{aligned}$$

Conversely, let P be any projector. Then $\mathrm{tr}\,[P(\rho-\sigma)] = \mathrm{tr}P(Q-S) \le \mathrm{tr}PQ \le \mathrm{tr}(Q) = D(\rho,\sigma)$. There is a closely related way of viewing the quantum trace distance which relates it more closely to the classical trace distance.

Theorem 9.1 *Let $\{E_m\}$ be a POVM, with $p_m \equiv \mathrm{tr}\rho E_m$ and $q_m \equiv \mathrm{tr}\sigma E_m$ as the probabilities of obtaining a measurement outcome labeled by m. Then*

$$D(\rho,\sigma) = \max_{\{E_m\}} D(p_m, q_m), \tag{9.23}$$

where the maximization is over all POVMs $\{E_m\}$

Proof
Note that

$$D(p_m, q_m) = \frac{1}{2}\sum_m |\mathrm{tr}(E_m(\rho-\sigma))|. \tag{9.24}$$

$$\left(\ \because\ (9.1)\ D(p_x,q_x) \equiv \frac{1}{2}\sum_x |p_x - q_x|\ \right)$$

We may write

$$\begin{aligned}|\mathrm{tr}(E_m(\rho-\sigma))| &= |\mathrm{tr}(E_m(Q-S))| &(9.25)\\ &\le \mathrm{tr}(E_m(Q+S)) &(9.26)\\ &\le \mathrm{tr}(E_m|\rho-\sigma|). &(9.27)\end{aligned}$$

Thus

$$\begin{aligned}D(p_m,q_m) &\le \frac{1}{2}\sum_m \mathrm{tr}(E_m|\rho-\sigma|) &(9.28)\\ &= \frac{1}{2}\mathrm{tr}|\rho-\sigma|\ \left(\ \because\ \sum_m E_m = I\ \right) &(9.29)\\ &= D(\rho,\sigma). &(9.30)\end{aligned}$$

Conversely, by choosing a measurement whose POVM elements include projetors onto the support for Q and S, we see that there exist measurements which give rise to probability distributions such that $D(p_m, q_m) = D(\rho,\sigma)$. □

If two density operators are close in trace distance, then any measurement performed on those quantum states will give rise to probability distributions which are close together in the classical sense of trace distance, giving a second interpretation of the trace distance between probability distributions arising from measurements performed on those quantum states.

The trace distance is a metric on the space of density operators.

$$D(\rho, \sigma) = 0$$

iff $\rho = \sigma$, and $D(.,.)$ is symmetric:

$$D(\rho, \sigma) = D(\sigma, \rho).$$

Moreover, the triangle inequality holds,

$$D(\rho, \tau) \leq D(\rho, \sigma) + D(\sigma, \tau). \tag{9.31}$$

From Equation (9.22) that there exists a project P such that

$$\begin{aligned}
D(\rho, \tau) &= \mathrm{tr}(P(\rho - \tau)) && (9.32)\\
&= \mathrm{tr}(P(\rho - \sigma)) + \mathrm{tr}(P(\sigma - \tau)) && (9.33)\\
&\leq D(\rho, \sigma) + D(\rho + \tau). && (9.34)\\
&\quad (\ \because \ \ \mathrm{tr}(P(\rho - \sigma)) \leq \mathrm{tr}(P_{\rho-\sigma}(\rho - \sigma)) \ ) && (9.35)
\end{aligned}$$

The most interesting result is that no physical process ever increase the distance between two quantum states.

Theorem 9.2 (Trace-preserving quantum operations are contractive) *Suppose $\mathcal{E}$ is a trace-preserving quantum operation. Let ρ and σ be density oprators. Then*

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma). \tag{9.36}$$

Proof
We see that

$$\begin{aligned}
D(\rho, \sigma) &= \frac{1}{2}\mathrm{tr}|\rho - \sigma| && (9.37)\\
&= \frac{1}{2}\mathrm{tr}|Q - S| \ \ (\ \because \rho - \sigma = Q - S, \ |\rho - \sigma| = Q + S \ ) && (9.38)\\
&= \frac{1}{2}\mathrm{tr}(Q) + \frac{1}{2}\mathrm{tr}(S) && (9.39)\\
&= \frac{1}{2}\mathrm{tr}(\mathcal{E}(Q)) + \frac{1}{2}\mathrm{tr}(\mathcal{E}(S)) \ \ (\because \ \text{trace preserving} \ ) && (9.40)\\
&= \mathrm{tr}(\mathcal{E}(Q)) \ \ (\ \because \mathrm{tr}(\rho - \sigma) = \mathrm{tr}(Q - S) = 0 \ ) && (9.41)\\
&\geq \mathrm{tr}(P\mathcal{E}(Q)) && (9.42)\\
&\geq \mathrm{tr}(P(\mathcal{E}(Q) - \mathcal{E}(S))) && (9.43)\\
&= \mathrm{tr}(P(\mathcal{E}(\rho) - \mathcal{E}(\sigma))) && (9.44)\\
&= D(\mathcal{E}(\rho), \mathcal{E}(\sigma)), && (9.45)
\end{aligned}$$

which completes the proof. □

eg 1) Gallery
eg 2) If we 'cover up' parts of two quantum states then we can show that the distance between those two states is never increased. If we take quantum state ρ^{AB} and σ^{AB} of a composite system AB then the distance between $\rho^A = \mathrm{tr}_\mathrm{B}\left[\rho^{AB}\right]$ and $\sigma^A = \mathrm{tr}_\mathrm{B}\left[\sigma^{AB}\right]$ is never more than the distance between ρ^{AB} and σ^{AB},

$$D(\rho^A, \sigma^A) \leq D(\rho^{AB}, \sigma^{AB}). \tag{9.46}$$

In many applications we want to estimate the trace distance for a mixture of inputs. Such estimates are greatly aided by the following theorem:

Theorem 9.3 (Strong convexity of the trace distance) *Let $\{p_i\}$ and $\{q_i\}$ be probability distributions over the same index set, and ρ_i and σ_i be density operators, also with indices from the same index set. Then*

$$D\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) \le D(p_i, q_i) + \sum_i p_i D(\rho_i, \sigma_i), \tag{9.47}$$

where $D(p_i, q_i)$ is the classical trace distance between the probability distribution $\{p_i\}$ and $\{q_i\}$.

Proof
By Equation(9.22) there exists a projector P such that

$$\begin{aligned}
& D\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) && (9.48)\\
= & \sum_i p_i \mathrm{tr}(P\rho_i) - \sum_i q_i \mathrm{tr}(P\sigma_i) && (9.49)\\
= & \sum_i \mathrm{tr}(P(\rho_i - \sigma_i)) + \sum_i (p_i - q_i)\mathrm{tr}(P\sigma_i) && (9.50)\\
& (\because \pm\sum_i \mathrm{tr}(P\sigma_i)) \\
\le & \sum_i p_i D(\rho_i, \sigma_i) + D(p_i, q_i) && (9.51)\\
& (\because \mathrm{tr}(P(\rho_i - \sigma_i) \le \mathrm{tr}(P_{\rho_i - \sigma_i}(\rho_i - \sigma_i) = \sum_i p_i D(\rho_i, \sigma_i)) \\
& (\because \sum_i (p_i - q_i)\mathrm{tr}(P\sigma_i) = \sum_{i\in S}(p_i - q_i) \le \max_S \sum_{i\in S}(p_i - q_i) = D(p_i, q_i)) && (9.52)
\end{aligned}$$

9.2.2 Fidelity

A second measure of distance between quantum states is the *fidelity.* The dielity is not a metric on density operators, but we will see that it does give rise to a useful metric. The fidelity of states ρ and σ is defined to be

$$F(\rho, \sigma) \equiv \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \tag{9.53}$$

eg1) When $[\rho, \sigma] = 0$,

$$\rho = \sum_i r_i |i\rangle\langle i|; \quad \sigma = \sum_i s_i |i\rangle\langle i|. \tag{9.54}$$

$$\begin{aligned}
F(\rho, \sigma) &= \mathrm{tr}\sqrt{\sum_i r_i s_i |1\rangle\langle 1|} && (9.55)\\
&= \mathrm{tr}\left(\sum_i \sqrt{r_i s_i}|i\rangle\langle i|\right) && (9.56)\\
&= \sqrt{r_i s_i} && (9.57)\\
&= F(r_i, s_i). && (9.58)
\end{aligned}$$

That is, when ρ and σ commute the quantum fidelity $F(\rho, \sigma)$ reduces to the classical fidelity $F(r_i, s_i)$ between the eigenvalues distributions r_i and s_i of ρ and σ.

eg2) The fidelity between a pure state $|\psi\rangle$ and an arbitrary state ρ. We see that

$$\begin{aligned} F(|\psi\rangle, \rho) &= \mathrm{tr}\sqrt{\langle\psi|\rho|\psi\rangle|\psi\rangle\langle\psi|} && (9.59)\\ &= \sqrt{\langle\psi|\rho|\psi\rangle}. && (9.60)\end{aligned}$$

That is, the fidelity is equal to the square root of the overlap between $|\psi\rangle$ and ρ. This is an important result which we will make use of often.

There's no clear geometric interpretation for the fidelity between two states of a qubit.

However, the fidelity does satisfy many of the same properties as the trace distance. For example, it is invariant under unitary transformations:

$$F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma). \tag{9.61}$$

There is also a useful characterization of the fidelity analogous to the characterization (**??**9.22)) for the trace distance.

Theorem 9.4 (Uhlmann's theorem) *Suppose ρ and σ are states of a quantum system Q. Introduce a second quantum system R which is a copy of Q. Then*

$$F(\rho, \sigma) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle|, \tag{9.62}$$

where the maximization is over all purification $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ into RQ.

Lemma 9.5 *Let A be any operator, and U unitary. Then*

$$|\mathrm{tr}(AU)| \le \mathrm{tr}|A|, \tag{9.63}$$

with equality being attained by choosing $U = V^\dagger$, where $A = |A|V$ is the polar docomposition of A.

Proof
Equality is clearly attained under the conditins stated. Observe that

$$|\mathrm{tr}(AU)| = |\mathrm{tr}(|A|VU)| = |\mathrm{tr}|A|^{1/2}|A|^{1/2}VU|. \tag{9.64}$$

The Cauchy–Schwarz inequality for the Hilbert-Schmidt inner product gives:

$$|\mathrm{tr}(AU)| \le \sqrt{\mathrm{tr}|A|\mathrm{tr}(U^\dagger V^\dagger |A| VU)} = \mathrm{tr}|A|, \tag{9.65}$$

which is completes the proof. *Proof*
Uhlmann's theorem
Fix orthonormal bases $|i_R\rangle$ and $|i_Q\rangle$ in systems R and Q. Because R and Q are of the same dimension, the index i may be assumed to run over the same set of values. Define $|m\rangle = \sum_i |i_R\rangle|i_Q\rangle$. Let $|\psi\rangle$ be any purification of ρ. Then the Schmidt decomposition and a moment's thought should convince you that

$$|\psi\rangle = (U_R \otimes \sqrt{\rho}U_Q)|m\rangle, \tag{9.66}$$

for some unitary operators U_R and U_Q. Similarly, if $|\phi\rangle$ is any purification of σ then there exist unitary operators V_R and V_Q such that

$$|\phi\rangle = (V_R \otimes \sqrt{\sigma}V_Q)|m\rangle. \tag{9.67}$$

Then, the inner product is

$$|\langle\psi|\phi\rangle| = \langle m|\Big({U_R}^\dagger V_R \otimes {U_Q}^\dagger \sqrt{\rho}\sqrt{\sigma}V_Q\Big)|m\rangle|. \tag{9.68}$$

Using

$$\mathrm{tr}(A^\dagger B) = \langle m|(A \otimes B)|m\rangle,$$

$$|\langle\psi|\phi\rangle| = |\mathrm{tr}\Big(V_R{}^\dagger U_R U_Q{}^\dagger \sqrt{\rho}\sqrt{\sigma}V_Q\Big)|. \tag{9.69}$$

Setting $U \equiv V_Q V_R{}^\dagger U_R U_Q{}^\dagger$ we see that

$$|\langle\psi|\phi\rangle| = |\mathrm{tr}(\sqrt{\rho}\sqrt{\sigma}U)|. \tag{9.70}$$

By Lemma 9.5,

$$|\langle\psi|\phi\rangle| \leq \mathrm{tr}|\sqrt{\rho}\sqrt{\sigma}| = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{9.71}$$

To see that equality may be attained, suppose $\sqrt{\rho}\sqrt{\sigma} = |\sqrt{\rho}\sqrt{\sigma}|V$ is the polar decomposition of $\sqrt{\rho}\sqrt{\sigma}$.

Properties of the fidelity are more easily proved using Uhlmann's formula in many instances.

*eg)*Uhlmann's formula shows that the fidelity is

- symmetric in its inputs, $F(\rho,\sigma) = F(\sigma,\rho)$
- bounded between 0 and 1, $0 \leq F(\rho,\sigma) \leq 1$.

On the other hand original formula of the fidelity is sometimes useful as a means for understanding properties of the fidelity. For instance, we see that $F(\rho,\sigma) = 0$ iff ρ and σ have support on orthogonal subspaces. $\rightarrow$ perfectly distinguishable. The relation between the quantum fidelity and the classical fidelity.

$$F(\rho,\sigma) = \min_{E_m} F(p_m, q_m), \tag{9.74}$$

where the minimum is over all POVMs $\{E_m\}$, and $p_m \equiv \mathrm{tr}(\rho E_m)$, $q_m \equiv \mathrm{tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVMs $\{E_m\}$. To see that this is true, note that

$$\begin{aligned} F(\rho,\sigma) &= \mathrm{tr}(\sqrt{\rho}\sqrt{\sigma}U) & (9.75)\\ &= \sum_m \mathrm{tr}(\sqrt{\rho}\sqrt{E_m}\sqrt{E_m}\sqrt{\sigma}U).\ \ (\ \because \sum_m E_m = I\ ) & (9.76)\end{aligned}$$

The Cauchy-Schwarz inequality and some simple algebra gives

$$\begin{aligned} F(\rho,\sigma) &\leq \sqrt{\mathrm{tr}(\rho E_m)\mathrm{tr}(\sigma E_m)} & (9.77)\\ &= F(p_m, q_m), & (9.78)\end{aligned}$$

which establishes that

$$F(\rho,\sigma) \leq F(p_m, q_m). \tag{9.79}$$

To see that equality can be obtained in this inequality, we need to find a POVM $\{E_m\}$ such that the Cauchy–Schwarz inequality is satisfied with equality for each term in the sum, that is,

$$\sqrt{E_m}\sqrt{\rho} = \alpha_m\sqrt{E_m}\sqrt{\sigma}U$$

for some complex number α_m. But

$$\sqrt{\rho}\sqrt{\sigma} = \sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

, so for invertible ρ,

$$\sqrt{\sigma}U = \rho^{-1/2}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{9.80}$$

We find the equality conditions are that

$$\sqrt{E_m}(I - \alpha_m M) = 0 \tag{9.81}$$

$$\begin{aligned} (\ \because\ \sqrt{E_m}\sqrt{\rho} &= \alpha_m\sqrt{E_m}\rho^{-1/2}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\\ \sqrt{E_m} &= \alpha\sqrt{E_m}M\ ) \end{aligned}$$

Three important properties of the trace distance

- the metric property
- contractivity
- strong convexity

Analogous properties all hold for the fidelity.

The fidelity is not a metric; however, there is a simple way of turning the fidelity into a metric. Uhlmann's theorem tells us that the fidelity between two states is equal to the maximum inner product between purifications of those states. We define the *angle* between states ρ and σ by

$$A(\rho, \sigma) \equiv \arccos F(\rho, \sigma). \tag{9.82}$$

Obviously the angle is non-negative, symmetric in its inputs, and is equal to zero iff $\rho = \sigma$. $\rightarrow$ a metric.

We prove the triangle inequality using Uhlmann's theorem and some obvious facts about vectors in three dimensions. Let *othercat*, $|\psi\rangle$, $|\gamma\rangle$ be purifications of σ, ρ, and τ respectively such that

$$\begin{aligned} F(\rho, \sigma) &= \langle\psi|\phi\rangle & (9.83) \\ F(\sigma, \tau) &= \langle\phi|\gamma\rangle, & (9.84) \end{aligned}$$

and $\langle\psi|\gamma\rangle$ is real and positive. ($\rightarrow$ **imagine Schmidt decomposition**) It is clear that (**image generic pure states on a sphere**)

$$\arccos(\langle\psi|\gamma\rangle) \leq A(\rho, \sigma) + A(\sigma, \tau). \tag{9.85}$$

But by Uhlmann's theorem, $F(\rho, \tau) \geq \langle\psi|\gamma\rangle$, so $A(\rho, \tau) \leq \arccos(\langle\psi|\gamma\rangle)$. Combining with the previous inequality gives the triangle inequality,

$$A(\rho, \tau) \leq A(\rho, \sigma) + A(\sigma, \tau). \tag{9.86}$$

Qualitively, the fidelity behaves like an 'upside-down' version of the trace distance, decreasing as two states become more distinguishable, and increasing as they become less distinguishable. Therefore

- trace distance: *contractivity* $\rightarrow$ fidelity: *monotonicity*
- trace distance: *strong concavity* $\rightarrow$ fidelity: *strong convexity*

Theorem 9.6 (Monotonicity of the fidelity) *Suppose $\mathcal{E}$ is a trace-preserving quantum operation. Let ρ and σ be density operators. Show that*

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma) \tag{9.87}$$

Proof
Let $|\psi\rangle$ and $|\phi\rangle$ be purification of ρ and σ into a joint system RQ such that $F(\rho, \sigma) = |\langle\psi|phi\rangle|$. Introduce a model environment E for the quantum operation, $\mathcal{E}$, which starts in a pure state $|0\rangle$, and interacts with the quantum system Q via a unitary interaction U. Note that $U|\psi\rangle|0\rangle$ is a purification of $\mathcal{E}(\rho)$, and $U|\phi\rangle|0\rangle$ is a purification of $\mathcal{E}(\sigma)$. By Ulhmann's theorem it follows that

$$\begin{aligned} F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &\geq |\langle\psi|\langle 0|U^\dagger U|\phi\rangle|0\rangle| & (9.88) \\ &= |\langle\psi|\phi\rangle| & (9.89) \\ &= F(\rho, \sigma), & (9.90) \end{aligned}$$

establishing the property we set out to prove. $\square$

Theorem 9.7 (Strong concavity of the fidelity) *Let p_i and q_i be probability distributions over the same index set, and ρ_i and σ_i density operators also indexed by the same index set. Then*

$$F\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) \geq \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i). \tag{9.91}$$

Proof
Let $|psi_i\rangle$ and $|\phi_i\rangle$ be purification of ρ_i and σ_i chosen such that $F(\rho_i, \sigma_i) = \langle\psi_i|\phi_i\rangle$. Introduce an ancillary system which has orthonormal basis states $|i\rangle$ corresponding to the index set i for the probability distributions. Define

$$|\psi\rangle \equiv \sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle; \quad |\phi\rangle \equiv \sum_i \sqrt{q_i}|\phi_i\rangle|i\rangle. \tag{9.92}$$

Note that $|\psi\rangle$ is a purification of $\sum_i p_i\rho_i$ and $|\phi\rangle$ is a purification of $\sum_i q_i\sigma_i$,

$$\left(cf. \quad \begin{array}{ll} \mathrm{tr}_{\mathrm{R}}(|\psi\rangle\langle\psi|) & = \sum_i p_i|\psi_i\rangle\langle\psi_i| \\ & = \sum_i p_i\rho_i \end{array} \right)$$

$$=$$

so by Uhlmann's formula,

$$F\left(\sum_i p_i\rho_i, \sum_i q_i\sigma_i\right) \geq |\langle\psi|\phi\rangle| = \sum_i \sqrt{p_i q_i}\langle\psi_i|\phi_i\rangle = \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i). \tag{9.93}$$

which establishes the result we set out to prove. □

9.2.3 Relationships between distance measures

The trace distance and the fidelity may be considered to be *equivalent* measures of distance for many applications.

In the case of pure states, the trace distance and the fidelity are completely equivalent to one another. Consider the trace distance between two pure states $|a\rangle$ and $|b\rangle$. Using Gram-Schmidt procedure we may find orthonormal states $|0\rangle$ and $|1\rangle$ such that

$$|a\rangle = |0\rangle; \quad |b\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle.$$

Note that $F(|a\rangle, |b\rangle) = |\cos\theta|$. Furthermore,

$$\begin{aligned} D(|a\rangle, |b\rangle) &= \frac{1}{2}\mathrm{tr}\left|\begin{bmatrix} 1-\cos^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta \end{bmatrix}\right| && (9.94) \\ &= |\sin\theta| && (9.95) \\ &= \sqrt{1 - F(|a\rangle, |b\rangle)^2}. && (9.96) \\ &= (\ F^2 + D^2 = 1 \ \ cf. \text{ PRA } \mathbf{50} \text{ 1047}) && (9.97) \end{aligned}$$

This relationship at the level of pure states can be used to deduce a relationship at the level of mixed states. Let ρ and σ be any two quantum states, and let $|\psi\rangle$ and $|\phi\rangle$ be purifications chosen such that $F(\rho, \sigma) = |\langle\psi|\phi\rangle| = F(|\psi\rangle, |\phi\rangle)$. We see that

$$\begin{aligned} D(\rho, \sigma) &\leq D(|\psi\rangle, |\phi\rangle) && (9.98) \\ &= \sqrt{1 - F(\rho, \sigma)^2}. && (9.99) \end{aligned}$$

Thus, if the fidelity between two states is close to one, it follows that the states are also close in trace distance. The converse is also true. To see this, let $\{E_m\}$ be a POVM such that

$$F(\rho, \sigma) = \sum_m \sqrt{p_m q_m}, \tag{9.100}$$

where $p_m \equiv \mathrm{tr}(\rho E_m)$, $q_m \equiv \mathrm{tr}(\sigma E_m)$ are the probabilities for obtaining outcome m for the states *rho* and σ, respectively. Observe first that

$$\begin{aligned} \sum_m (\sqrt{p_m} - \sqrt{q_m})^2 &= \sum_m p_m + \sum_m q_m - 2F(\rho, \sigma) \\ &= 2(1 - F(\rho, \sigma)). \end{aligned}$$

However, it is also true that $|\sqrt{p_m} - \sqrt{q_m}| \leq |\sqrt{p_m} + \sqrt{q_m}|$, so

$$\begin{aligned} \sum_m (\sqrt{p_m} - \sqrt{q_m})^2 &= \sum_m |\sqrt{p_m} - \sqrt{q_m}||\sqrt{p_m} + \sqrt{q_m}| && (9.101) \\ &= \sum_m |p_m - q_m| && (9.102) \\ &= 2D(p_m, q_m) && (9.103) \\ &\leq 2D(\rho, \sigma). \quad (\because (9.23)) && (9.104) \end{aligned}$$

Thus

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma). \tag{9.105}$$

Summarizing, we have

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{9.106}$$

The implication is that the trace distance and the fidelity are qualitatively equivalent measures of closeness for quantum states. Indeed, for many purposes it does not matter whether the trace distance or the fidelity is used to quantify distance, since results about one may be used to deduce equivalent results about the other.