8.3 Examples of quantum noise and quantum operations

8.3.1 Trace and partial trace

One of the main uses of the quantum operations formalism is to describe the effects of measurement.

Quantum operations formalism can be used to describe both the *probability of getting a particular outcome from a measurement on a quantum system*, and also *the state change n the system effected by the measurement.*

The trace map $\rho \to \mathrm{tr}\,[\rho]$– the simplest operation related to measurement.

H_Q : any input Hilbert space, spanned by an orthonormal basis $|1\rangle \ldots |d\rangle$
H'_Q : any dimensional ouput space, spanned by the state $|0\rangle$

$$\mathcal{E}(\rho) \equiv \sum_{i=1}^{d} |0\rangle\langle i|\rho|i\rangle\langle 0|, \tag{8.83}$$

so that $\mathcal{E}$ is a quantum operation, by Theorem 8.1. ($E_i = |0\rangle\langle i|$). The quantum operation is identical to the trace function ($|0\rangle$: unimportant)

The partial trace is a quantum operation.

QR: a joint system, and we wish to trace out system R.
$|j\rangle$: a basis for system R.

Define a linear operator $E_i : H_{QR} \to H_Q$ by

$$E_i \left(\sum_j \lambda_j |q_j\rangle |j\rangle \right) \equiv \lambda_i |q_i\rangle, \tag{8.84}$$

where λ_j are complex numbers, and $|q_j\rangle$ are arbitrary states of system Q. Define $\mathcal{E}$ to be the quantum operation with operation elements $\{E_i\}$, that is,

$$\mathcal{E}(\rho) \equiv \sum_i E_i \rho E_i{}^{\dagger}. \tag{8.85}$$

Bye Theorem 8.1, this is a quantum operation from system QR to system Q. Notice that

$$\mathcal{E}(\rho \otimes |j\rangle\langle j'|) = \rho\delta_{j,j'} = \mathrm{tr}\,[\rho \otimes |j\rangle\langle j'|] \tag{8.86}$$

where ρ is any Hermitian operator on the state space of system Q, and $|j\rangle$ and $|j'\rangle$ are members of the orthonormal basis for system R. By linearity of $\mathcal{E}$ and $\mathrm{tr_R}$, it follows that $\mathcal{E} = \mathrm{tr_R}$.

8.3.2 Geometric picture of single qubit quantum operations

The state of a single qubit can always be written in the Bloch epresentation,

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}, \tag{8.87}$$

where $\vec{r}$ is a three component real vector. Explicitly, this gives us

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix}. \tag{8.88}$$

An arbitrary trace-preserving quantum operation is equivalent to a map of the form

$$\vec{r} \xrightarrow{\mathcal{E}} \vec{r'} = M\vec{r} + \vec{c}, \tag{8.89}$$

where M is a 3×3 real matrix, and $\vec{c}$ is a constant vector. This is an *affine map*, mapping the Bloch sphere into itself.

Affine map

Any transformation preserving collinearity and ratios of distances. A affine map may also be thought of as a shearing transformation

$$\begin{aligned} F &: \mathbb{R}^n \to \mathbb{R}^n \\ F(\vec{p}) &= A\vec{p} + \vec{q} \end{aligned}$$

To see this, suppose the operators E_i generating the operator-sum representation for $\mathcal{E}$ are written in the form

$$E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik}\sigma_k \tag{8.90}$$

Then

$$M_{jk} = \sum_l \left[a_{lj}a^*_{lk} + a_{lj} * a_{lk} + \left(|\alpha_l|^2 - \sum_p a_{lp}a^*_{lp} \right) \delta_{jk} + i \sum_p \epsilon_{jkp}(\alpha_l a^*_{lp} - \alpha^*_l a_{lp}) \right] \tag{8.91}$$

$$c_k = 2i \sum_l \sum_{jp} \epsilon_{jpk} a_{lj} a^*_{lp}, \tag{8.92}$$

The meaning of the affine map is made clearer by considering the polar decomposition of the matrix M, $M = U|M|$, where U is unitary.
$M \to$ real
$|M| \to$ real and Hermitian (symmetric)
$U \to$ real (orthogonal), $U^T U = I$
Thus we may write

$$M = OS, \tag{8.93}$$

where O is a real orthogonal matrix with determinant 1, representing a proper rotation, and S is a real symmetric matrix. Equation(8.89) is just a deformation of the Bloch sphere along principal axes determined by S, followed by a proper rotation due to O, followed by a displacement due to $\vec{c}$.

8.3.3 Bit flip and phase flip channels

The *bit flip* channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1 - p$. It has operation elements

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p} X = \sqrt{1-p} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8.94}$$

$$\mathcal{E}(\rho) = \frac{1}{2} \left[I + r_x X + (2p-1) r_y Y + (2p-1) r_z Z \right]$$

The contraction of the Bloch sphere illustrated in figure 8.8 cannot increase the norm of the Bloch vector, and therefore we can immediately conclude that tr $\left[\rho^2\right]$ can only ever decrease for the bit flip channel.

The *phase flip* channel has operation elements

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p} Z = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.95}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\left[I + (2p-1)r_x X + (2p-1)r_y Y + r_z Z\right]$$

If $p = \frac{1}{2}$?

Using the freedom in the operator-sum representation this operation may be written [*cf.* (8.66), (8.67)]

$$\rho \rightarrow \mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1, \tag{8.96}$$

where $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$, which corresponds to a measurement of the qubit in the $|0\rangle$, $|1\rangle$ basis, with the result of the measurement unknown. Using the above presentation it is easy to see that the corresponding map on the Bloch sphere is given by

$$(r_x, r_y, r_z) \rightarrow (0, 0, r_z) \tag{8.97}$$

The *bit-phase flip* channel has operation elements

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p} Y = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.98}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\left[I + (2p-1)r_x X + r_y Y + (2p-1)r_z Z\right]$$

8.3.4 Depolarizing channel

The *depolarizing channel* is an important type of quantum noise. Imagine we take a single qubit, and with probability p that qubit is *depolarized.* That is, it is replaced by the completely mixed state, $I/2$. With probability $1-p$ the qubit is left untouched. The state of the quantum system after this noise is

$$\mathcal{E}(\rho) = \frac{pI}{2} + (1-p)\rho \tag{8.99}$$

$$\mathcal{E}(\rho) = \frac{1}{2} I + \frac{1}{2}(1-p)\vec{r} \cdot \vec{\sigma}$$

Figure (8.12) : The top line of the circuit is the input to the depolarizing channel, while the bottom two lines are an 'environment' to simulate the channel.

The idea is that the third qubit, initially a mixture of the state $|0\rangle$ with probability $1-p$ and state $|1\rangle$ with probability p acts as a control for whether or not the completely mixed state $I/2$ stored in the second qubit is swapped into the first qubit.

The form (8.99) is not in the operator-sum representation. For arbitrary ρ,

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} \tag{8.100}$$

and then substitute for $I/2$ into (8.99) we arrive at the equation

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right), \tag{8.101}$$

showing that the depolarizing channel has operation elements $\{\sqrt{1-3p/4}I, \sqrt{p}X/2, \sqrt{p}Y/2, \sqrt{p}Z/2\}$. Note, incidentally, that it is frequently convenient to parameterize the depolarizing channel in different ways, such as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z). \tag{8.102}$$

8.3.5 Amplitude damping

An important application of quantum operations is the description of *energy dissipation* – effects due to loss of energy from a quantum system.

- The dynamics of an atom which is spontaneously emitting a photon.
- A spin system at high temperature approach equilibrium with its environment.
- The state of a photon in an interferometer or cavity when it is subject to scattering and attenuation.

$\rightarrow$ the general behavior of all of them is well characterized by a quantum operation known as *amplitude damping.*

We have a single optical mode containing the quantum state $a|0\rangle + b|1\rangle$. The scattering of a photon from this mode can be modeled by thinking of inserting a partially silvered mirror, a beamsplitter, in the path of the photon. This beamsplitter allows the photon to couple to another single optical mode, according to the unitary transformation $B = \exp[\theta(a^\dagger b - ab^\dagger)]$. The output after the beamsplitter, assuming the environment starts out with no photons, is simply $B|0\rangle(a|0\rangle + b|1\rangle) = a|00\rangle + b(\cos\theta|01\rangle + \sin\theta|10\rangle)$. Tracing over the environment gives us the quantum operation

$$\mathcal{E}_{AD}(\rho) = E_0\rho E_0 + E_1\rho E_1, \tag{8.103}$$

where $E_k = \langle k|B|0\rangle$ are

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{8.104}$$

The operation elements for amplitude damping. $\gamma = \sin^2\theta$ can be thought of as the probability of losing a photon.

The E_1 operation changes a $|1\rangle$ stat into a $|0\rangle$ state, corresponding to the physical process of losing a quantum of energy to the environment. E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; physically, this happens because a quantum of energy was not lost to the environment, and thus the environment now perceives it to be more likely that the system is in the $|0\rangle$, rather than the $|1\rangle$ state.

A general characteristic of a quantum operation is the set of state that are left invariant under the operation. (*eg.* the phase flip channel : $\hat{z}$ axis invariance) In the case of amplitude damping, only the ground state $|0\rangle$ is left invariant. That is a natural consequence of our modeling the environment as starting in the $|0\rangle$ state, as if it were at zero temperature.

What quantum operation describe the effect of dissipation to an environment at finite temperature? $\mathcal{E}_{GAD}$: *generalized amplitude damping*, is defined for single qubits by the operation elements

$$E_0 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \tag{8.116}$$

$$E_1 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \tag{8.117}$$

$$E_2 = \sqrt{1-p}\begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix} \tag{8.118}$$

$$E_3 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix} \tag{8.119}$$

where the stationary state

$$\rho_\infty = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \tag{8.120}$$

satisfies $\mathcal{E}_{GAD} = \rho_\infty$. Generalized amplitude damping describes the 'T_1' relaxation processes due to coupling of spins to their surrounding lattice, a large system which is in thermal equilibrium at a temperature often much higher than the spin temperature.

The effect of amplitude damping in the Bloch representation as the Bloch vector transformation

$$(r_x, r_y, r_z) \to \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma)\right). \tag{8.122}$$

When γ is replaced with a time-varying function like $1-e^{-t/T_1}$, as is often the case for real physical processes.

Similarly, generalized amplitude damping performs the trasformation

$$(r_x, r_y, r_z) \to \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma(2p-1) + r_z(1-\gamma)\right). \tag{8.123}$$

Amplitude damping and generalized amplitude damping differ only in the location of the fixed point of the flow; the final state is along the $\hat{z}$ axis, at the point $(2p-1$, which is a mixed state.

8.3.6 Phase damping

A noise process that is uniquely quantum mechanical, which is describes the loss of quantum information without loss of energy, is *phase damping.*

- A photon scatters randomly as it travels through a waveguide
- Electronic states in an atom are perturbed upon interacting with distant electrical charges.

The energy eigenstates of a quantum system do not change as a function of time, but do accumulate a phase which is not precisely known, partial information about this quantum phase – the *relative* phases between the energy eigenstates - is lost.

A very simple model:
Suppose we have a qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ upon which the rotation operation $R_z(\theta)$ is applied, where the angle of rotation θ is random. (a *phase kick*) Let us assume that the phase kick angle θ is well represented as a random variable which has a Gaussian distribution with mean 0 and variance 2λ.

The output state from this process is given by the density matrix obtained from averaging over θ,

$$\rho = \frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} R_z(\theta)|\psi\rangle\langle\psi|{R_z}^\dagger(\theta) e^{-\theta^2/4\lambda} d\theta \tag{8.124}$$

$$= \begin{bmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{bmatrix}. \tag{8.125}$$

The random phase kicking causes the expected value of the off-diagonal elements of the density matrix to decay exponentially to zero with time. That is a characteristic result of phase damping.

Consider an interaction between two harmonic oscillators with interaction Hamiltonian

$$H = \chi a^\dagger a(b + b^\dagger). \tag{8.126}$$

Letting $U = \exp(-iH\Delta t)$, considering only the $|0\rangle$ and $|1\rangle$ states of the a oscillator as our system, and taking the environment oscillator to initially be $|0\rangle$, we find that tracing over the environment gives the operation elements $E_k = \langle k_b|U|0_b\rangle$, which are

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} \tag{8.127}$$

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \tag{8.128}$$

where $\lambda = 1 - \cos^2(\chi\Delta)$ can be interpreted as the probability that a photon from the system has been scattered (without loss of energy). As was the case for amplitude damping, E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; unlike amplitude damping, however, the E_1 operation destroys $|0\rangle$ and reduces the amplitude of the $|1\rangle$ state, and does not change it into a $|0\rangle$.

By applying the unitary freedom of quantum operations,

$$\tilde{E}_0 = \sqrt{\alpha}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.129}$$

$$\tilde{E}_1 = \sqrt{1-\alpha}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{8.130}$$

where $\alpha = (1+\sqrt{1-\lambda})/2$. Thus the phase damping quantum operation is ***exactly*** the same as the phase flip channel.

Phase damping is often referred to as a 'T_2' (or 'spin-spin') relaxation process, where $e^{-t/2T_2} = \sqrt{1-\lambda}$.

Historically, phase damping was a process that was almost always thought of as resulting from a random phase kick or scattering process. It was not until the connection tot the phase flip channel was discovered that quantum error-correction was developed, since it was thought that phase errors were continuous and couldn't be described as ad discrete process.