8.2 Quantum operations

8.2.1 Overview

The quantum operations formalism is a general tool for describing the evolution of quantum systems in a wide variety of circumstances, including stochastic changes to quantum states, much as Markov processes describe stochastic changes to classical states. Just as a classical state is described by a vector of probabilities, we shall describe quantum states in terms of the density operator ρ. Similar to (8.3), quantum states transform as

$$\rho' = \mathcal{E}(\rho) \tag{8.4}$$

The map $\mathcal{E}$ in this equation is a *quantum operation*.

eg)

- Unitary transformation $\mathcal{E}(\rho) = U\rho U^\dagger$
- Measurement $\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$

The quantum operation captures the dynamic change to a state which occurs as the result of some physical process; ρ is the initial state before the process, and $\mathcal{E}(\rho)$ is the final state after the process occurs, possibly up to some normalization factor.

We shall develop *three* separate ways of understanding quantum operations.

1. The fist method is based on the idea of studying dynamics as the result of an interaction between a system and an environment, much as classical noise was described in Section 8.1. This method is concrete and easy to relate to the real world. However, it suffers from the drawback of not being mathematically convenient.
2. The second method of understanding quantum operations, completely equivalent to the first, overcomes this mathematical inconvenience by providing a powerful mathematical representation for quantum operations known as *operator-sum representation.* This method is rather abstract, but is very useful for calculations and theoretical work.
3. The third approach to quantum operations, equivalent to the other two, is via a set of physically motivated axioms that we would expect a dynamical map in quantum mechanics to satisfy. The advantage of this approach is that it is exceedingly general. However, it does not offer the calculational convenience of the second approach, nor the concrete nature of the first.

8.2.2 Environments and quantum operations

Consider the unitary transform as a box into which the input state enters and from which the output exits. For our purpose, the interior workings of the box are not of concern to us; it could be implemented by a quantum circuit, or by some Hamiltonian system, or anything else.

A natural way to describe the dynamics of an open system : principal system $\leftrightarrow$ environment (which together form a closed quantum system)

Suppose we have a system in state ρ, which is sent into a box which is coupled to an environment. In general the final state of the system, $\mathcal{E}(\rho)$, may not be related by a unitary transformation to the initial state ρ. We assume that the system–environment input state is a product the environment, and thus we perform a partial trace over the environment to obtain the reduced state of the system alone:

$$\mathcal{E}(\rho) = \mathrm{tr}_{\mathrm{env}}\left[U(\rho \otimes \rho_{\mathrm{env}})U^\dagger\right] \tag{8.6}$$

Of course, if U does not involve any interaction with the environment, the $\mathcal{E}(\rho) = \tilde{U}\rho\tilde{U}^\dagger$ where $\tilde{U}$ is the part of U which acts on the system alone.

An important assumption is made in this definition – the system and the environment start in a product state. In general, of course, this is not true. Quantum systems interact constantly with their environments, building up correlations.($\rightarrow$ Heat exchange)

However, in many cases of practical interest it is reasonable to assume that the system and its environment start out a product state. Ideally, the correlations will be completely destroyed, leaving the system in a pure state. Even if this is not the case, we shall see later that the quantum operations formalism can even describe quantum dynamics when the system and environment do not start out n a product state.

How can U be specified if the environment has nearly infinite degrees of freedom? It turns out that in order for this model to properly describe any possible transformation $\rho \rightarrow \mathcal{E}(\rho)$, if the principal system has a Hilbert space of d dimensions, then it suffices to model the environment as being in a Hilbert space of no more than d^2 dimensions. It turns out not to be necessary for the environment to start out in a mixed state.

eg) The two qubit quantum circuit in which U is a controlled-NOT gate, with the principal system the control qubit, and the environment initially in the state $\rho_{\text{env}} = |0\rangle\langle 0|$ as the target qubit. Inserting into Equation (8.6), it is easily seen that

$$\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1, \tag{8.7}$$

where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projection operators.Intuitively, this dynamics occurs because the environment stays in the $|0\rangle$ state only when the system is $|0\rangle$; otherwise the environment is flipped to the state $|1\rangle$.

It is convenient to generalize the definition somewhat to allow different input and output spaces.

eg) A single qubit(A) is prepared in an unknown state ρ. A three-level quantum system ('qutrit') labelled B is prepared in some standard state $|0\rangle$, and then interacts with system A via a unitary interaction U, causing the joint system to evolve into the state $U(\rho \otimes |0\rangle\langle 0|)U^\dagger$. We then discard system A, leaving system B in some final state ρ'. The quantum operation $\mathcal{E}$ describing this process is

$$\mathcal{E}(\rho) = \rho' = \text{tr}_A(U(\rho \otimes |0\rangle\langle 0|)U^\dagger) \tag{8.8}$$

Notice that $\mathcal{E}$ maps density operators of the input system, A, to density operators of the output system, B. Most of our discussion of quantum operations below is concerned with quantum operations 'on' some system A, that is, they map density operators of system A to density operators of system A. The quantum operation $\mathcal{E}$ defining this process simply maps ρ to ρ'.

8.2.3 Operator-sum representation

The operator-sum representation : essentially re-statement of Equation (8.6) explicitly in terms of operators on the principal system's Hilbert space alone.

Let $|e_k\rangle$ be an orthonormal basis for the state space of the environment, and let $\rho_{\text{env}} = |e_0\rangle\langle e_0|$ be the initial state of the environment. There's no loss of generality in assuming that the environment starts in a pure state. (Section 2.5). Equation (8.6) can be rewritten as

$$\begin{aligned} \mathcal{E}(\rho) &= \sum_k \langle e_k|U\left[\rho \otimes |e_0\rangle\langle e_0|\right]U^\dagger|e_k\rangle & (8.9) \\ &= \sum_k E_k\rho E_k^\dagger, & (8.10) \end{aligned}$$

where $E_k \equiv \langle e_k|U|e_0\rangle$ is an operator n the state space of the principal system. Equation (8.10) is known as the operator-sum representation of $\mathcal{E}$. The operators $\{E_k\}$ are known as *operator elements* for the quantum operation $\mathcal{E}$. The operator-sum representation is important.

The operation elements satisfy an important constraint known as the *completeness relation*, analogous to the completeness relation for evolution matrices in the description of classical noise.

Classical case The completeness relation arises from the requirement that probability distributions be normalized to one.

Quantum case The completeness relation arises from the analogous requirement that the trace of $\mathcal{E}(\rho)$ be equal to one

$$\begin{aligned} 1 &= \operatorname{tr}(\mathcal{E}(\rho)) && (8.11)\\ &= \operatorname{tr}\left(\sum_k E_k \rho E_k^\dagger\right) && (8.12)\\ &= \operatorname{tr}\left(\sum_k E_k^\dagger E_k \rho\right). && (8.13) \end{aligned}$$

Since this relationship is true for all ρ it follows that we must have

$$\sum_k E_k^\dagger E_k = I. \tag{8.14}$$

This equation is satisfied by quantum operations which are *trace-preserving*. There are also non-trace-preserving quantum operations, for which $\sum_k E_k^\dagger E_k \leq I$, but they describe processes in which extra information about what occurred in the process is obtained by measurement.

Maps $\mathcal{E}$ of the form of (8.10) for which $\sum_k E_k^\dagger E_k \leq I$ provide our second definition of a quantum operation. (including cases for non-trace-preserving operations)

The operator-sum representation is important:

- it gives us an intrinsic means of characterizing the dynamics of the principal system. It describes the dynamics of the principal system without having to explicitly consider properties of the environment; all that we need to know is bundled up into the operators E_k, which act on the principal system alone.
- This simplifies calculations and often provides considerable theoretical insight.
- Many different environmental interactions may give rise to the same dynamics on the principal system.

The properties of the operator-sum representation:

1. What's a physical interpretation, in terms of the operation elements E_k.
2. How an operator-sum representation can be determined for any open quantum system (measurement).
3. How to construct a model open quantum system for any operator-sum representation(modelling).

Physical interpretation of the operator-sum representation

Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary transformation U has been applied. Applying the principle of implicit measurement, we see that such a measurement affects only the state of the environment, and does not change the state of the principal system.

Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\rho_k \propto \mathrm{tr_E}(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|) = \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle \tag{8.18}$$

$$= E_k \rho E_k^\dagger. \tag{8.19}$$

Normalizing ρ_k,

$$\rho_k \frac{E_k \rho E_k^\dagger}{\mathrm{tr}(E_k \rho E_k^\dagger)} \tag{8.20}$$

we find the probability of outcome k is given by

$$p(k) = \mathrm{tr}(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|) \tag{8.21}$$

$$= \mathrm{tr}(E_k \rho E_k^\dagger) \tag{8.22}$$

Thus

$$\mathcal{E}(\rho) = \sum_k p(k)\rho_k = \sum_k E_k \rho E_k^\dagger. \tag{8.23}$$

This gives us a beautiful physical *interpretation* of what is going on in a quantum operation with operation elements $\{E_k\}$. The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by $E_k \rho E_k^\dagger / \mathrm{tr}\left[E_k \rho E_k^\dagger\right]$, with probability $\mathrm{tr}\left[E_k \rho {E_k}^\dagger\right]$. In this sense, it is very similar to the concept of noisy communication channels used in classical information theory; in this vein, we shall sometimes refer to certain quantum operations which describe quantum noise processes as being noisy quantum channels.

eg) Figure 8.4

Suppose we choose the state $|e_k\rangle = |0_E\rangle$ and $|1_E\rangle$. This can be interpreted as doing a measurement in the computational basis of the environment qubit. Doing such a measurement does not change the state of the principal system. The controlled-NOT may be expanded as

$$U = |0_P 0_E\rangle\langle 0_P 0_E| + |0_P 1_E\rangle 0_P \bar{1}_E + |1_P 1_E\rangle\langle 1_P 0_E| + |1_P 0_E\rangle\langle 1_P 1_E|. \tag{8.24}$$

$$E_0 = \langle 0_E|U|0_E\rangle = |0_P\rangle\langle 0_P| \tag{8.25}$$

$$E_1 = \langle 1_E|U|0_E\rangle = |1_P\rangle\langle 1_P| \tag{8.26}$$

and therefore

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1, \tag{8.27}$$

Measurement and the operator-sum representation

How to determine an operator-sum representation for an open quantum system $\rightarrow$ the unitary system-environment transformation operation U and a basis of state $|e_k\rangle$ for the environment, the operation elements are

$$E_k \equiv \langle e_k|U|e_0\rangle. \tag{8.28}$$

It is possible to extend this result even further by allowing the possibility that a measurement is performed on the combined system-environment after the unitary interaction, allowing the acquisition of information about the quantum state. $\rightarrow$ **non-trace-preserving quantum operations**, that is, maps $\mathcal{E}(\rho) \sum_k E_k \rho {E_k}^\dagger$ such that $\sum_k {E_k}^\dagger E_k \leq I$.

Suppose:
$\rho \rightarrow$ initial state (Q: principal system, E: environment)
The joint state of the system is initially

$$\rho^{QE} = \rho \otimes \sigma. \tag{8.29}$$

After some U (: unitary interaction) a projective measurement (P_m) is performed n the joint system.

The final state of QE is given by

$$\frac{P_m U(\rho \otimes \sigma) U^\dagger P_m}{\mathrm{tr}\left[P_m U(\rho \otimes \sigma) U^\dagger P_m\right]}, \tag{8.30}$$

given that measurement outcome m occurred. Tracing out E we see that the final state of Q alone is

$$\frac{\mathrm{tr_E}\left[P_m U(\rho \otimes \sigma) U^\dagger P_m\right]}{\mathrm{tr}\left[P_m U(\rho \otimes \sigma) U^\dagger P_m\right]}. \tag{8.31}$$

This representation of the final state involves the initial state σ of the environment, the interaction U and the measurement operators P_m. Define a map

$$\mathcal{E}_m(\rho) \equiv \mathrm{tr_E}\left[P_m U(\rho \otimes \sigma) U^\dagger P_m\right], \tag{8.32}$$

so the final state of Q alone is $\mathcal{E}_m(\rho)/\mathrm{tr}\left[\mathcal{E}_m(\rho)\right]$. Note that $\mathrm{tr}\left[\mathcal{E}_m(\rho)\right]$ is the probability of outcome m of the measurement occurring.

Let $\sigma = \sum_j q_j |j\rangle\bar{j}$ be an ensemble decomposition for σ. Introduce an orthonormal basis $|e_k\rangle$ for the system E. Note that

$$\begin{aligned}
\mathcal{E}_m(\rho) &= \sum_{jk} q_j \mathrm{tr_E}\left[|e_k\rangle\langle e_k P_m U(\rho \otimes |j\rangle\langle j|) U^\dagger P_m |e_k\rangle\langle e_k||\right] && (8.33)\\
&= \sum_{jk} E_{jk}\rho E_{jk}{}^\dagger, && (8.34)
\end{aligned}$$

where

$$E_{jk} \equiv \sqrt{q_j}\langle e_k|P_m U|j\rangle. \tag{8.35}$$

This equation is a generalization of Equation (8.10), and gives an explicit means for calculating the operators appearing in an operator-sum representation for $\mathcal{E}_m$. The quantum operation $\mathcal{E}_m$ can be thought of as defining a kind of measurement process generalizing the description of measurements (Ch. 2)

System-environment models for any operator-sum representation

Given a set of operators $\{E_k\}$ is there some reasonable *model environmental system and dynamics* which give rise to a quantum operation with those operation element?

For any trace-preserving or non-trace-preserving quantum operation, $\mathcal{E}$, with operation elements $\{E_k\}$, there exists a model environment, E, starting in a pure state $|e_0\rangle$, and model dynamics specified by a unitary operator U and projector P onto E such that

$$\mathcal{E}(\rho) = \mathrm{tr_E}\left[PU(\rho \times |e_0\rangle\langle e_0|)U^\dagger P\right]. \tag{8.36}$$

Suppose first that $\mathcal{E} \rightarrow$ a trace-preserving quantum operation (elements $\{E_k\}$) satisfying the completeness relation $\sum_k E_k{}^\dagger E_k = I$, U?

Let $|e_k\rangle$ be an orthonormal basis set for E, in one-to-one correspondence with the index k for the operators E_k. We are trying to find a *model* environment giving rise to a dynamics described by the operation elements $\{E_k\}$. Define an operator U which has the following action on states of the form $|\psi\rangle|e_0\rangle$,

$$U|\psi\rangle|e_0\rangle \equiv \sum_k E_k|\psi\rangle|e_k\rangle, \tag{8.37}$$

where $|e_0\rangle$ is just some standard state of the model environment. For arbitrary states $|\psi\rangle$ and $|\phi\rangle$ of the principal system,

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \sum_k \langle\psi|E_k{}^\dagger E_k|\phi\rangle = \langle\psi|\phi\rangle \tag{8.38}$$

by the completeness relation. Thus the operator U can be extended to a unitary operator acting on the entire state space of the joint system.

$$\mathrm{tr}_{\mathrm{E}}\left[U(\rho \times |e_0\rangle\langle e_0|)U^\dagger\right] = \sum_k E_k \rho {E_k}^\dagger, \tag{8.39}$$

so this model provides a realization of the quantum operation $\mathcal{E}$ with operation elements $\{E_k\}$.

Non-trace-preserving quantum operation → A more interesting generalization of this construction is the case of a set of quantum operations $\{\mathcal{E}_m\}$ corresponding to possible outcomes from a measurement, so the quantum operation $\sum_m \mathcal{E}_m$ is trace-preserving, since the probabilities of the distinct outcomes sum to one, $1 = \sum_m p(m) = \mathrm{tr}\,[(\sum_m \mathcal{E}_m)(\rho)]$ for all possible inputs ρ.

Box 8.1: Mocking up a quantum operation

8.2.4 Axiomatic approach to quantum operations

We are going to switch to a different viewpoint now, where we try to write down physically motivated axioms which we expect quantum operation to obey. → more abstract.

First, we're going to forget everything we've learned about quantum operations, and start over by defining quantum operations according to a set of axioms, which wel" justify on physical grounds. That done, we'll prove that a map $\mathcal{E}$ satisfies these axioms iff it has an operator-sum representation, thus providing the missing link between the abstract axiomatic formulation, and our earlier discussion.

We *define* a quantum operator $\mathcal{E}$ as a map from the set of density operators of the input space Q_1 to the set of density operators for the output space Q_2, with the following three axiomatic properties:

- **A1**: First, $\mathrm{tr}\,[\mathcal{E}(\rho)]$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state. Thus, $0 \leq \mathrm{tr}\,[\mathcal{E}(\rho)] \leq 1$ for any state ρ.
- **A2**: Second, $\mathcal{E}$ is a *convex-linear map* on the set of density matrices, that is, for probabilities $\{p_i\}$,

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i) \tag{8.43}$$

- **A3**: Third, $\mathcal{E}$ is a completely positive map. That is, if $\mathcal{E}$ maps density operators of system Q_1 to density operators of system Q_2, then $\mathcal{E}(A)$ must be positive for any positive operator A. Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive for any positive operator A on the combined system RQ_1, where $\mathcal{I}$ denotes the identity map on system R.

It is convenient to make the convention that $\mathcal{E}$ does not necessarily preserve the trace property of density matrices, that $\mathrm{tr}\,[\rho] = 1$. Rather, we make the convention that $\mathcal{E}$ is to be defined in such a way that $\mathrm{tr}\,[\mathcal{E}(\rho)]$ is equal to the probability of the measurement outcome described by $\mathcal{E}$ occurring.

eg) Suppose that we are doing a projective measurement in the computational basis of a single qubit. Then two quantum operations are used to describe this process, defined by $\mathcal{E}_0(\rho) \equiv |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) \equiv |1\rangle\langle 1|\rho|1\rangle\langle 1|$. Notice that the probabilities of the respective outcomes are correctly given by $\mathrm{tr}\,[\mathcal{E}_0(\rho)]$ and $\mathrm{tr}\,[\mathcal{E}_1(\rho)]$. With this convention the correctly normalized final quantum state is therefore

$$\frac{\mathcal{E}(\rho)}{\mathrm{tr}\,[\mathcal{E}(\rho)]} \tag{8.44}$$

The first property is one of mathematical convenience. In the case where the process is deterministic, that is, no measurement is taking place, this reduces to the requirement that $\mathrm{tr}\,[\mathcal{E}(\rho)] = 1 = \mathrm{tr}(\rho)$, for all ρ. ($\rightarrow$ non-trace-preserving) A physical quantum operation is one that satisfies the requirement that probabilities never exceed 1, $\mathrm{tr}\,[\mathcal{E}(\rho)] \leq 1$.

The second property stems from a physical requirement on quantum operations.

$$\rho = \sum_i p_i \rho_i$$

Then we would expect that the resulting state,

$$\frac{\mathcal{E}(\rho)}{\mathrm{tr}\,[\mathcal{E}(\rho)]} = \frac{\mathcal{E}(\rho)}{p(\mathcal{E})}$$

corresponds to a random select from the ensemble $\{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/\mathrm{tr}\,[\mathcal{E}(\rho_i)]\}$, where $p(i|\mathcal{E})$ is the probability that the state prepared was ρ_i, given that the process represented by $\mathcal{E}$ occurred. Thus, we demand that

$$\mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}\,[\mathcal{E}(\rho_i)]}, \tag{8.45}$$

where $p(\mathcal{E}) = \mathrm{tr}\,[\mathcal{E}(\rho)]$ is the probability that the process described by $\mathcal{E}$ occurs on input of ρ. By Bayes' rule

$$(i|\mathcal{E}) = p(\mathcal{E}|i) \frac{p_i}{p(\mathcal{E})} = \frac{\mathrm{tr}\,[\mathcal{E}(\rho_i)]\, p_i}{p(\mathcal{E})} \tag{8.46}$$

so (8.45) reduces (8.43).

The third property also originates from an important physical requirement, that not only must $\mathcal{E}(\rho)$ be a valid density matrix (up to normalization) so long as ρ is valid, but furthermore, if $\rho = \rho_{RQ}$ is the density matrix of some joint system of R and Q, if $\mathcal{E}$ acts only on Q, then $\mathcal{E}(\rho_{RQ})$ must still result in a valid density matrix of the joint system. Formally, suppose we introduce a second system R. Let $\mathcal{I}$ denote the identity map n system R. Then the map $\mathcal{I} \otimes \mathcal{E}$ must take positive operators to positive operators.

Box 8.2: Complete positivity versus positivity

The transpose operation on a single qubit $\rightarrow$ preserves positivity of a single qubit.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{8.47}$$

This map preserves positivity of a single qubit. <u>However</u>, suppose that qubit is part of a two qubit system initially in the entangled state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{8.48}$$

and the transpose operation is applied to the first of these two qubits. Then the density operator of the system after the dynamics has been applied is

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.49}$$

This operator has eigenvalues $1/2$, $1/2$, $1/2$, and $-1/2$, so this is not a valid density operator. Thus the transpose operation is an example of a positive map which is not completely positive, that is, it preserves the positivity of operators on the principal system, but does not continue to preserve positivity when applied to systems which contain the principal system as a subsystem.

Theorem 8.1 *The map $\mathcal{E}$ satisfies axioms **A1**, **A2**, and **A3** iff*

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i{}^\dagger, \tag{8.50}$$

for some set of operators $\{E_i\}$ which map the input Hilbert space to the output Hilbert space, and $\sum_i E_i{}^\dagger E_i \leq I$.

Proof
Suppose $\mathcal{E}(\rho) = \sum_i E_i \rho E_i{}^\dagger$.
$\mathcal{E} \rightarrow$ Linear, so to check that $\mathcal{E}$ is a quantum operation we need only prove that it is completely positive.

$A \rightarrow$ a positive operator acting on RQ.
$|\psi\rangle \rightarrow$ some state of RQ.
def. $|\phi_i\rangle \equiv (I_R \otimes E_i{}^\dagger)|\psi\rangle$, we have

$$\begin{aligned} \langle\psi|(I_R \otimes E_i)A(I_R \otimes E_i{}^\dagger)|\psi\rangle &= \langle\phi_i|A|\phi_i\rangle && (8.51) \\ &\geq 0, && (8.52) \end{aligned}$$

by the positivity of the operator A. It follows that

$$\langle\psi|(\mathcal{I} \otimes \mathcal{E})(A)|\psi\rangle = \sum_i \langle\phi_i|A|\phi_i\rangle \geq 0, \tag{8.53}$$

and thus for any positive operator A, the operator $(\mathcal{I} \otimes \mathcal{E})(A)$ is also positive. The requirement $\sum_i E_i{}^\dagger E_i \leq I$ ensures that probabilities are less than or equal to 1.

Suppose next that $\mathcal{E}$ satisfies axioms **A1**, **A2**, and **A3**. Our aim will be to find an operator-sum representation for $\mathcal{E}$.

R : the same dimension as the original quantum system Q.
$|i_R\rangle$, $|i_Q\rangle$: orthonormal bases for R and Q.

Define a joint state $|\alpha\rangle$ of RQ by

$$|\alpha\rangle \equiv = \sum_i |i_R\rangle|i_Q\rangle. \tag{8.54}$$

The state $|\alpha\rangle$ is a maximally entangled state of the systems R and Q. Next, we define an operator σ on the state space of RQ by

$$\sigma \equiv (\mathcal{I}_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|). \tag{8.55}$$

We may think of this as the result of applying the quantum operation $\mathcal{E}$ to one half of a maximally entangled state of the system RQ. $\rightarrow$ the operator σ completely specifies the quantum operator $\mathcal{E}$. That is, to know how $\mathcal{E}$ acts on an arbitrary state of Q, it is sufficient to know how it acts on a single maximally entangled state of Q with another system.

Let $|\psi\rangle = \sum_j c_j |j_Q\rangle$ be any state of Q. Define a corresponding state $|\tilde{\psi}\rangle$ of system R by the equation

$$|\tilde{\psi}\rangle = \sum_j c_j^* |j_R\rangle. \tag{8.56}$$

Notice that

$$\begin{aligned} \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle &= \langle\tilde{\psi}| \left(\sum_{ij} |i_R\rangle\langle i_R| \otimes \mathcal{E}(|i_Q\rangle\langle i_Q|) \right) |\tilde{\psi}\rangle && (8.57) \\ &= \sum_{ij} c_i c_j^* \mathcal{E}(|i_Q\rangle\langle i_Q|) && (8.58) \\ &= \mathcal{E}(|\psi\rangle\langle\psi|). && (8.59) \end{aligned}$$

Let $\sigma = \sum_i |s_i\rangle\langle s_i|$ be some decomposition of σ, where the vector $|s_i\rangle$ need not be normalized. Define a map

$$E_i(|\psi\rangle) \equiv \langle\tilde{\psi}|s_i\rangle. \tag{8.60}$$

This map is a linear map, so E_i is a linear operator on the state space Q. Furthermore we have

$$\begin{aligned}\sum_i E_i|\psi\rangle\langle\psi|E_i{}^\dagger &= \sum_i \langle\tilde{\psi}|s_i\rangle\langle s_i|\tilde{\psi}\rangle && (8.61)\\ &= \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle && (8.62)\\ &= \mathcal{E}(|\psi\rangle\langle\psi|). && (8.63)\end{aligned}$$

Thus

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i|\psi\rangle\langle\psi|E_i{}^\dagger \tag{8.64}$$

for all pure states, $|\psi\rangle$, of Q. By convex-linearity it follows that

$$\mathcal{E}\rho = \sum_i E_i\rho E_i{}^\dagger \tag{8.65}$$

in general The condition $\sum_i E_i{}^\dagger E_i \le I$ follows immediately from axiom **A1** identifying the trace of $\mathcal{E}(\rho)$ with a probability. □

Freedom in the operator-sum representation

We have seen that the operator-sum representation provides a very general description of the dynamics of an open quantum system. Is it a *unique* description?

Consider quantum operations $\mathcal{E}$ and $\mathcal{F}$ acting on a single qubit with the operator-sum representations $\mathcal{E}(\rho) = \sum_k E_k\rho E_k{}^\dagger$ and $\mathcal{F}(\rho) = \sum F_k\rho F_k{}^\dagger$, where the operation elements for $\mathcal{E}$ and $\mathcal{F}$ are defined by

$$E_1 = \frac{I}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad E_2 = \frac{Z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{8.66}$$

and

$$F_1 = |0\rangle\langle 0| = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \qquad F_2 = |1\rangle\langle 1| = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}. \tag{8.67}$$

These appear to be very different quantum operations. What is interesting is that $\mathcal{E}$ and $\mathcal{F}$ are *actually the same quantum operation.* ($\because F_1 = (E_1 + E_2)/\sqrt{2}$, $F_2 = (E_1 - E_2)/\sqrt{2}$) Thus,

$$\mathcal{F}(\rho) = \mathcal{E}(\rho) \tag{8.70}$$

The operation elements appearing in an operator-sum representation for a quantum operation are *not* unique. A random applying either I or Z has the same dynamics of a projective measurement in the $\{|0\rangle, |1\rangle\}$ basis. $\rightarrow$ Two very different physical process give rise to exactly the same dynamics for the principal system.

Understanding the freedom in operator-sum representation:

- gives us more insight into how different physical processes can give rise to the same system dynamics
- is crucial to a good understanding of quantum error-correction.

From (8.10),

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_k \langle e_k|U\left[\rho\otimes|e_0\rangle\langle e_0|\right]U^\dagger|e_k\rangle\\ &= \sum_k E_k\rho E_k^\dagger,\end{aligned}$$

where $E_k = \langle e_k|U|e_0\rangle$. Suppose that we supplement the interaction U with an additional unitary action U' on the environment alone. Clearly this does not change the state of the principal system. [even they are entangled] We obtain:

$$\begin{aligned}
F_k &= \langle e_k|(I \otimes U')|e_0\rangle && (8.71)\\
&= \sum_j \left[I \otimes \langle e_k|U'|e_j\rangle\right] \langle e_j|U|e_0\rangle && (8.72)\\
&= \sum_j U'_{kj} E_j, && (8.73)
\end{aligned}$$

U'_{kj} are the matrix elements of U' with respect to the basis $|e_k\rangle$

Theorem 8.2 (Unitary freedom in the operator-sum representation) *Suppose* $\{E_1, \dots, E_m\}$ *and* $\{F_1, \dots, E_n\}$ *are operation elements giving rise to quantum operations* $\mathcal{E}$ *and* $\mathcal{F}$*, respectively. By appending zero operators to the shorter list of operation elements we may ensure that* $m = n$*. The* $\mathcal{E} = \mathcal{F}$ *iff there exist complex numbers* u_{ij} *such that* $E_i = \sum_j u_{ij} F_j$*, and* u_{ij} *is an* m *by* m *unitary matrix.*

proof
Recall the theorem 2.6 *Unitary freedom in the ensemble for density matrices.* Two sets of vectors $|\psi_i\rangle$ and $|\phi_i\rangle$ generate the same operator iff

$$|\psi_i\rangle = \sum_j u_{ij}|\phi_i\rangle, \tag{8.74}$$

where u_{ij} is a unitary matrix of complex numbers, and $|\psi_i\rangle$ and $|\phi_i\rangle$ have the same number of elements. Suppose $\{E_i\}$ and $\{F_i\}$ are two sets of operation elements for the same quantum operation,$\sum_i E_i \rho {E_i}^\dagger = \sum_j F_j \rho {F_j}^\dagger$ for all ρ. Define

$$\begin{aligned}
|e_i\rangle &\equiv \sum_k |k_R\rangle(E_i|k_Q\rangle) && (8.75)\\
|f_j\rangle &\equiv \sum_k |k_R\rangle(F_j|k_Q\rangle) && (8.76)
\end{aligned}$$

Recall $\sigma \equiv (\mathcal{I}_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|)$, from which it follows that $\sigma = \sum_i |e_i\rangle\langle e_i| = \sum_j |f_j\rangle\langle f_j|$, and thus there exits unitary u_{ij} such that

$$|e_i\rangle = \sum_j u_{ij}|f_j\rangle. \tag{8.77}$$

But for arbitrary $|\psi\rangle$ we have

$$\begin{aligned}
E_i|\psi\rangle &= \langle\tilde{\psi}|e_i\rangle && (8.78)\\
&= \sum_j u_{ij}\langle\tilde{\psi}|f_j\rangle && (8.79)\\
&= \sum_j u_{ij} F_j|\psi\rangle && (8.80)
\end{aligned}$$

Thus

$$E_i = \sum_j u_{ij} F_j. \tag{8.81}$$

Conversely, supposing E_i and F_j are related by a unitary transformation of the form $E_i = \sum_j u_{ij} F_j \rightarrow$ the quantum operation with operation elements $\{E_i\}$ is the same as the quantum operation with operation elements $\{F_i\}$. $\square$

Theorem 8.3 *All quantum operation $\mathcal{E}$ on a system of Hilbert space dimension d can be generated by an operator-sum representation containing at most* d^2 *elements,*

$$\mathcal{E}(\rho) = \sum_{k=1}^{M} E_k \rho {E_k}^{\dagger}, \tag{8.82}$$

where $1 \leq M \leq d^2$.