

4.2 Single qubit operations

A single qubit is a vector $|\psi\rangle = a|0\rangle + b|1\rangle$ parameterized by two complex numbers satisfying $|a|^2 + |b|^2 = 1$. Operations on a qubit must preserve this norm, and thus are described by 2×2 unitary matrices. Of these, some of the most important are the Pauli matrices;

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Three other quantum gates will play a large part in what follows, the Hadamard gate (H), phase gate (S), and $\pi/8$ gate (T):

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; T \equiv \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}.$$

A couple of useful algebraic facts to keep in mind are that $H = (X + Z)/\sqrt{2}$ and $S = T^2$. The reason that T gate is called the $\pi/8$ gate is for unimportant global phase T is equal to a gate which has $\exp(\mp i\pi/8)$ appearing on its diagonals.

A single qubit in the state $a|0\rangle + b|1\rangle$ can be visualized as a point (θ, ϕ) on the unit sphere, where $a = \cos(\theta/2)$, $b = e^{i\phi} \sin(\theta/2)$, and a can be taken to be real because the overall phase of the state is unobservable. This is called Bloch sphere representation, and the vector $(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ is called the Bloch vector.

The Pauli matrices give rise to three useful classes of unitary matrices when they are exponentiated, the *rotation operators* about the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes, defined by the equations:

$$\begin{aligned} R_x(\theta) &\equiv e^{-i\theta X/2} = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} X = \begin{bmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \\ R_y(\theta) &\equiv e^{-i\theta Y/2} = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Y = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \\ R_z(\theta) &\equiv e^{-i\theta Z/2} = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \end{aligned}$$

If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions then we generalize the previous definitions by defining a rotation by θ about the $\hat{n}$ axis by the equation

$$R_{\hat{n}} \equiv \exp(-i\theta \hat{n} \cdot \vec{\sigma}/2) = \cos \left(\frac{\theta}{2} \right) I - i \sin \left(\frac{\theta}{2} \right) (n_x X + n_y Y + n_z Z),$$

An arbitrary unitary operator on a single qubit can be written in many ways as a combination of rotations, together with global phase shifts on the qubit.

Theorem 4.1: (Z-Y decomposition for a single qubit) Suppose U is a unitary operation on a single qubit. Then there exist real numbers α , β , γ and δ such that

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

Corollary 4.2: Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A , B , C on a single qubit such that $ABC = I$ and $U = e^{i\alpha}AXBXC$, where α is some overall phase factor.

Proof

In the notation of Theorem 4.1, set $A \equiv R_z(\beta)R_y(\gamma/2)$, $B \equiv R_y(-\gamma/2)R_z(-(\delta+\beta)/2)$ and $C \equiv R_z((\delta-\beta)/2)$. Note that

$$ABC = R_z(\beta)R_y\left(\frac{\gamma}{2}\right)R_y\left(-\frac{\gamma}{2}\right)R_z\left(-\frac{\delta+\beta}{2}\right)R_z\left(\frac{\delta-\beta}{2}\right) = I.$$

Since $X^2 = I$, $XYX = -Y$, and $XR_y(\theta)X = R_y(-\theta)$, we see that

$$XBX = XR_y\left(-\frac{\gamma}{2}\right)XXR_z\left(-\frac{\delta+\beta}{2}\right)X = R_y\left(\frac{\gamma}{2}\right)R_z\left(\frac{\delta+\beta}{2}\right).$$

Thus

$$\begin{aligned} AXBXC &= R_z(\beta)R_y\left(\frac{\gamma}{2}\right)R_y\left(\frac{\gamma}{2}\right)R_z\left(\frac{\delta+\beta}{2}\right)R_z\left(\frac{\delta-\beta}{2}\right) \\ &= R_z(\beta)R_y(\gamma)R_z(\delta). \end{aligned}$$

Thus $U = e^{i\alpha}AXBXC$ and $ABC = I$, as required. $\square$