2.6 EPR and the Bell inequality

A compelling example of an essential difference between quantum and classical physics.

For an object, we assume that the physical properties of that object have an existence independent of observation. That is, measurements merely act to reveal such physical properties.

QM: an unobserved particle does not posses physical properties that exist independent of observation. Rather, such physical properties arise as a consequence of measurements performed upon the system.

EPR were interested in what they termed 'elements of reality'. Their belief was that any such element of reality must be represented in any complete physical theory. The goal of the argument was to show that quantum mechanics is not a complete phsical theory, by identifying elements of reality that were not included in quantum mechanics.

The way they attempted to do was by introducing what they claimed was a sufficient condition for a physical property to be an element of reality, namely, that it be possible to predict with certainty the value that property will have, immediately before measurement.

Box 2.7: Anti-correlation in the EPR experiment The prepared two qubit state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

The measurement of observable $\vec{v}\cdot\vec{\sigma}$ on each qubit will get $+1$ or -1 for each qubit. It turns out that no matter what choice of $\vec{v}$ we make, the results of the two measuerments are always opposite to one another.
Suppose $|a\rangle$ and $|b\rangle$ are the eigenstates of $\vec{v}\cdot\vec{\sigma}$. Then

$$\begin{aligned} |0\rangle &= \alpha|a\rangle + \beta|b\rangle \\ |1\rangle &= \gamma|a\rangle + \delta|b\rangle. \end{aligned}$$

Substituting we obtain

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\frac{|ab\rangle - |ba\rangle}{\sqrt{2}}$$

But $\alpha\delta - \beta\gamma$ is the determinant of the unitary matrix, and thus is equal to a phase factor $e^{i\theta}$ for some real θ. Thus

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|ab\rangle - |ba\rangle}{\sqrt{2}}$$

As a result, if a measuerment of $\vec{v}\cdot\vec{\sigma}$ is performed on both qubits, then we can see that a result of $+1(-1)$ on the first qubit implies a result of $-1(+1)$ on the second qubit.

eg) Consider, an entangled pair of qubits belonging to Alice and Bob, respectively:

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Alice performs a measurement of spin along the $\vec{v}$ axis. Suppose Alice recives the result $+1$. Bob will measuer -1 on his qubit if he also measures sping along the $\vec{v}$ axis.

Above result seems to correspond to an element of reality, by the EPR criterion, and should be represented in any complete physical theory. **However** standard quantum mechanics merely tells one how to caculate the probabilities of the respective measurement outcomes if $\vec{v} \cdot \vec{\sigma}$ is measured. Standard quantum meachanics certainly does not include any fundamental element intended to represent the value of $\vec{v} \cdot \vec{\sigma}$, for all unit vector $\vec{v}$.

The goal of EPR was to show that quantum mechanics is incomplete, by demonstrating that quantum mechanics lacked some essential 'element of reality', by their criterion. They hoped to force a return to a more classical view of the world, one in which systems could be ascribed properties which existed independently of measurements performed on those systems.

EPR defeated nearly 30 years after its publishing, by invalidation of *Bell's Inequality.* Bell's inequality is *not* a result about quantum mechanics. To obtain Bell's inequality, we're going to do a thought(gedanken) experiment, which we will analyze using our common sense notions of how the world works(classically) – the sort of notions Einstein and his collaborators thought Nature ought to obey. After we have done the common sense analysis, we will perform a quantum mechanical analysis which we can show *is not consistent with the common sense analysis*

Thought Experiment:

- Charlie prepares to particles.
- Charlie sends one particle to Alice, and the second particle to Bob.
- Once Alice receives her particle, she performs a measurement on it using randomly one of two different measurements. These measurements are of physical properties which we shall labe P_Q and P_R, respectively. For simplicity, suppose that the measurements can each have one of two outcomes, $+1$ or -1. $P_Q \rightarrow Q$ and $P_R \rightarrow R$
- Similarly, $P_S \rightarrow S$ and $P_T \rightarrow T$, each taking value $+1$ or -1.
- Alice and Bob do their measurements *at the same time.* Therefore the measurement which performs cannot disturb the result of Bob's measurement, since physical influences cannot propagate faster than light.

We are going to do some simple algebra with the quantity $QS + RS + RT - QT$.

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T.$$

Because $R, Q = \pm 1$ it follows that either $(Q + R)S = 0$ or $(R - Q)T = 0$. In either case, it is easy to see from above equation that $QS + RS + RT - QT = \pm 2$.

Suppse that $p(q, r, s, t)$ is the probability that, before the emasurements are performed, the system is in a state where $Q = q, R = r, S = s, T = t$. These probabilities may depend on how Charlie performs his preparation, and on experimental noise. Letting $\mathrm{E}(\cdot)$ denote the mean value of a quantity, we have

$$\begin{aligned}\mathrm{E}(QS+RS+RT-QT) &= \sum_{qrst} p(q,r,s,t)(qs+rs+rt-qt) \\ &\leq \sum_{qrst} p(q,r,s,t)\times 2 \\ &= 2.\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{E}(QS+RS+RT-QT) &= \sum_{qrst} p(q,r,s,t)qs+\sum_{qrst} p(q,r,s,t)rs+\sum_{qrst} p(q,r,s,t)rt-\sum_{qrst} p(q,r,s,t)qt \\ &= \mathrm{E}(QS)+\mathrm{E}(RS)+\mathrm{E}(RT)-\mathrm{E}(QT).\end{aligned}$$

We obtain th *Bell inequality*,

$$\mathrm{E}(QS)+\mathrm{E}(RS)+\mathrm{E}(RT)-\mathrm{E}(QT)\leq 2$$

This result is also often known as the *CHSH inequality* after the initials of its four discoverers. It is part of a larger set of inequalities known generically as Bell inequalities.

Quantum mechanical analysis:

Charlie prepares a quantum system of two qubits in the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Alice and Bob perform measurements of the following observables:

$$\begin{aligned}Q &= Z_1 \\ R &= X_1 \\ S &= \frac{-Z_2-X_2}{\sqrt{2}} \\ T &= \frac{Z_2-X_2}{\sqrt{2}}.\end{aligned}$$

Simple calculations show that the average values for these observables are:

$$\langle QS\rangle = \frac{1}{\sqrt{2}};\langle RS\rangle = \frac{1}{\sqrt{2}};\langle RT\rangle = \frac{1}{\sqrt{2}};\langle QT\rangle = \frac{1}{\sqrt{2}};$$

Thus,

$$\langle QS\rangle+\langle RS\rangle+\langle RT\rangle-\langle QT\rangle = 2\sqrt{2}.$$

Above result violates the Bell inequality. And the experimental results denoted that the Bell inequality is not obeyed by Nature.

What does it mean? I means that one or more of the assumptions that went into the derivation of the Bell inequality must be incorrect.

There are two assumptions made in the proof of Bell inequality which are questionable:

1. The assumption that the physical properties P_Q, P_R, P_S, P_T have definite values Q, R, S, T which exist independent of observation. This is sometimes known as the assumption of *realism*.
2. The assumption that Alice performing her measurement does not influence the result of Bob's measurement. This is sometimes known as the assumption of *locality*.

These two assumptions together are known as the assumption of *local realism*. They are certainly intuitively plausible assumptions about how the world works, and they fit our everyday experience.

The world is *not* locally realistic. Bell's inequality points out to the conclusion that either or both of locality and realism must be dropped from our view of the world if are to develop a good intuitive understading of quantum mechanics.

Bell's inequality teaches us that entanglement is a fundamentally new resource in the world that goes essentially beyond classical resources.