2.5 The Schmidt decomposition and purifications

The study of composite quantum systems are at the heart of quantum computation and quantum information. Two additional tools of great value are the *Schmidt decomposition* and *purifications*.

theorem 2.7(**Schmidt decomposition**) $|\psi\rangle$: pure state of a composite system, AB. $\exists|i_A\rangle$ and $|i_B\rangle$: orthogonal sts. of A and B, respectively such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as *Schmidt co-efficients*.

This result is very useful.

If $|\psi\rangle$ is a pure st. of a composite system, AB, then by Schmidt decomposition

$$\rho^A = \sum_i \lambda^2 |i_A\rangle\langle i_A|$$

$$\rho^B = \sum_i \lambda^2 |i_B\rangle\langle i_B|$$

so, the eigenvalues of ρ^A and ρ^B are identical (λ^2) for both density operators. Many important properties of quantum system are completely determined by the eigenvalues of the reduced density operator of the system, so for a pure state of a composite system such properties will be the same for both sytems.

eg) two qubits, $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$

$$\begin{aligned} \rho^A &= \frac{1}{3}\left(2|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ \rho^B &= \frac{1}{3}\left(|0\rangle\langle 0| + 2|1\rangle\langle 1|\right) \\ \text{tr}\left(\left(\rho^A\right)^2\right) &= \frac{1}{9}\text{tr}\left(4|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ &= \frac{1}{9}\text{tr}\left(|0\rangle\langle 0| + 4|1\rangle\langle 1|\right) \\ &= \frac{5}{9} \end{aligned}$$

Proof
System A, B have state spaces of the same dimension.
$|j\rangle$, $|k\rangle$: any fixed orthonormal bases for systems A and B

$$|\psi\rangle = \sum_{jk} a_{jk}|j\rangle|k\rangle,$$

By the singular value decomposition, $a = udv$, where d is a diagonal matrix with non-negative elements, and u and v are unitary matrices

$$|\psi\rangle = \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle|k\rangle.$$

Defining $|i_A\rangle \equiv \sum_j u_{ji}|j\rangle, |i_B\rangle \equiv \sum_k v_{ik}|k\rangle$, and $\lambda = d_{ii}$

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$$

$|i_A\rangle$ and $|i_B\rangle$: *Schmidt bases* for A and B
λ_i : *Schmidt number* for the state $|\psi\rangle$. In some sense, Schmidt number is the 'amount' of entanglement between systems A and B. The Schmidt number is preserved under unitary transformations on system A or system B alone.

Purification
Suppose a system ρ^A and introduce another system, R, and define a pure state $|AR\rangle$ for the joint system AR such that $\rho^A = \mathrm{tr}_R(|AR\rangle\langle AR|)$. That is, the pure state $|AR\rangle$ reduces to ρ^A when we look at system A alone. Purification allows us to associate pure states with mixed states.

To prove that purification can be done for *any* state, we explain how to construct a system R and purification $|AR\rangle$ for ρ^A.

A system ρ^A has orthonormal decomposition $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$. To purify ρ^A, introduced R which has the same state space as system A, with orthonormal basis states $|i^R\rangle$, and define a pure state for the combined system

$$|AR\rangle \equiv \sum_i \sqrt{p_i}|i^A\rangle|i^R\rangle$$

We now calculate the reduced density operator for system A corresponding to the state $|AR\rangle$

$$\begin{aligned}\mathrm{tr}_R(|AR\rangle\langle AR|) &= \sum_{ij} \sqrt{p_i p_j}|i^A\rangle\langle j^A| \mathrm{tr}\left(|i^R\rangle\langle j^R|\right) \\ &= \sum_{ij} \sqrt{p_i p_j}|i^A\rangle\langle j^A|\delta_{ij} \\ &= \sum_i p_i |i^A\rangle\langle i^A| \\ &= \rho^A.\end{aligned}$$

Thus $|AR\rangle$is a purification of ρ^A.

The procedure used to purify a mixed state of system A is to define a pure state whose Schmidt basis for system A is just the basis in which the mixed state is diagonal, with the Schmidt coefficients being the square root of the eigenvalues of the density operator being purified.